Indifinite integral حل تمارين فصل ال

Find: (a) $\int \frac{x-3}{\sqrt{9x^2+4}}\,dx$ (b) $\int \frac{1+\cos(e^{-2x})}{e^{2x}}\,dx$

(c) $\int \frac{x^2}{\sqrt{16-x^6}}\,dx$ (d) $\int \cot x \cdot \ln(\sin x)\,dx$

Ans:

(a) $\int \frac{x-3}{\sqrt{9x^2+4}}\,dx = \frac{1}{3}\int \frac{x-3}{\sqrt{x^2+(\frac{2}{3})^2}}\,dx = \frac{1}{3}\left[\int \frac{x}{\sqrt{x^2+(\frac{2}{3})^2}}\,dx - 3\int \frac{dx}{\sqrt{x^2+(\frac{2}{3})^2}}\right]$

$= \frac{1}{3}\underbrace{\int \frac{1}{2}\cdot 2x\,(x^2+(\tfrac{2}{3})^2)^{-\frac{1}{2}}\,dx}_{J} - \int \frac{dx}{\sqrt{x^2+(\frac{2}{3})^2}} \longrightarrow \sinh^{-1}\left(\frac{3x}{2}\right)$

Put $u = x^2+(\frac{2}{3})^2 \Rightarrow du = 2x\,dx$

Then,

$J = \int \frac{1}{2}\cdot u^{-\frac{1}{2}}\,du = \frac{1}{\cancel{2}}\cdot \cancel{2}\cdot u^{\frac{1}{2}} = \sqrt{u}$. Then

$$\int \frac{x-3}{\sqrt{9x^2+4}}\,dx = \boxed{\frac{1}{3}\cdot\sqrt{x^2+\frac{4}{9}} - \sinh^{-1}\left(\frac{3x}{2}\right) + C}$$

(b) $I = \int \frac{1+\cos(e^{-2x})}{e^{2x}}\,dx = \int \frac{dx}{e^{2x}} + \int \frac{\cos(e^{-2x})}{e^{2x}}\,dx$

$= \int e^{-2x}\,dx + \int e^{-2x}\cdot\cos(e^{-2x})\,dx \longrightarrow$ Put $u = e^{-2x} \Rightarrow du = -2e^{-2x}\,dx$

Then

$\int e^{-2x}\cos(e^{-2x})\,dx = \frac{-1}{2}\int \cos(u)\,du$

$= \frac{-1}{2}\sin u = \frac{-1}{2}\sin(e^{-2x})$

$$= \boxed{\frac{-1}{2}e^{-2x} - \frac{1}{2}\sin(e^{-2x}) + C}$$

(c) $K = \int \frac{x^2}{\sqrt{16-x^6}}\,dx = \frac{1}{3}\int \frac{3x^2}{\sqrt{16-(x^3)^2}}\,dx$

Put $x^3 = u \Rightarrow 3x^2\,dx = du$, Then

$$K = \frac{1}{3}\int \frac{du}{\sqrt{(4)^2-u^2}} = \frac{1}{3}\sin^{-1}\left(\frac{u}{4}\right) + C = \boxed{\frac{1}{3}\sin^{-1}\left(\frac{x^3}{4}\right) + C}$$

(d) $J = \int \cot x \cdot \ln(\sin x)\,dx = \int \frac{\cos x}{\sin x}\ln(\sin x)\,dx$

Put $u = \ln(\sin x) \Rightarrow du = \frac{\cos x}{\sin x}\,dx$, Then

$$J = \int u\,du = \frac{1}{2}u^2 + C \Rightarrow \boxed{J = \frac{1}{2}[\ln(\sin x)]^2 + C}$$

Find: (a) $I = \int \frac{dx}{x^2-4x+7}$. (b) $J = \int \frac{dx}{2x^2-x-3}$

(c) $K = \int \frac{dx}{(x-1)\sqrt{x^2-2x-2}}$

Ans:

(a) $x^2-4x+7 = x^2-4x+4+7-4 = (x-2)^2+(\sqrt{3})^2$, Then

$$I = \int \frac{dx}{(x-2)^2+(\sqrt{3})^2} = \boxed{\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-2}{\sqrt{3}}\right)+C}$$

(b) using Partial fraction technique, Then

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$$J = \int \frac{1}{(2x-3)(x+1)}\,dx = \int\left[\frac{A}{(2x-3)} + \frac{B}{(x+1)}\right]dx$$

$A(x+1)+B(2x-3) = 1 \Rightarrow$ Put: $x=-1 \Rightarrow -5B = 1 \Rightarrow \boxed{B = \frac{-1}{5}}$

$x = \frac{3}{2} \Rightarrow \frac{5}{2}A = 1 \Rightarrow \boxed{A = \frac{2}{5}}$, Then

$$J = \frac{1}{5}\int \frac{2}{(2x-3)}\,dx - \frac{1}{5}\int \frac{dx}{x+1} = \boxed{\frac{1}{5}\ln\left[\frac{2x-3}{x+1}\right]+C}$$ #

(c) $$K = \int \frac{dx}{(x-1).\sqrt{x^2-2x+1-3}} = \int \frac{dx}{(x-1)\sqrt{(x-1)^2-(\sqrt{3})^2}} = \int \frac{du}{u.\sqrt{u^2-(\sqrt{3})^2}}$$

$$= \boxed{\frac{1}{\sqrt{3}}\sec^{-1}\left(\frac{(x-1)}{\sqrt{3}}\right)+C}$$

Find:

(a) $\int \frac{2x-1}{\sqrt{x^2+2x}}\,dx$	(b) $\int x\sqrt{x-1}\,dx$	(c) $\int \frac{dx}{3\sqrt{x}+1}$	(d) $\int \frac{dx}{\sqrt{x+2}-\sqrt{x}}$.

Ans:

(a) $$I = \int \frac{2x-1}{\sqrt{x^2+2x}}\,dx = \int \frac{2x-1}{\sqrt{x^2+2x+1-1}}\,dx$$

$$= \int \frac{2x.\,dx}{\sqrt{(x+1)^2-1}} - \int \frac{dx}{\sqrt{(x+1)^2-(1)^2}}$$ → $\cosh^{-1}(x+1)$

$$= \int \frac{2x+2-2}{\sqrt{(x+1)^2-1}}\,dx - \cosh^{-1}(x+1) = \int \frac{(2x+2)\,dx}{\sqrt{(x+1)^2-1}} - 2\int \frac{dx}{\sqrt{(x+1)^2-1}} - \cosh^{-1}(x+1)$$

$$= \boxed{2\sqrt{(x+1)^2-1} - 3\cosh^{-1}(x+1)+C}$$

(b) $I = \int x\sqrt{x-1}\,dx$.

Put $u^2 = x-1 \Rightarrow 2u\,du = dx$, Then

$$I = \int (u^2+1)\cdot u\cdot 2u\,du = 2\int (u^4+u^2)\,du = \frac{2}{5}u^5 + \frac{2}{3}u^2 + C$$

$$= \boxed{\frac{2}{5}\sqrt{(x-1)^5} + \frac{2}{3}\sqrt{(x-1)^3} + C}$$

(c) $I = \int \frac{dx}{3\sqrt{x}+1}$, Put $\sqrt{x} = u \Rightarrow x = u^2 \Rightarrow dx = 2u\,du$, Then

$$I = \int \frac{2u}{3u+1}\,du = \frac{2}{3}\int \frac{3u+1-1}{3u+1}\,du$$

$$= \frac{2}{3}\left[\int du - \frac{1}{3}\int \frac{3\,du}{3u+1}\right] = \frac{2}{3}u - \frac{1}{3}\ln|3u+1| + C$$

$$= \boxed{\frac{2}{3}\sqrt{x} - \frac{2}{9}\ln|3\sqrt{x}+1| + C}$$

(d) $I = \int \frac{dx}{\sqrt{x+2}-\sqrt{x}} * \frac{\sqrt{x+2}+\sqrt{x}}{\sqrt{x+2}+\sqrt{x}} = \int \frac{\sqrt{x+2}+\sqrt{x}}{x+2-x}\,dx$

$$= \frac{1}{2}\int \sqrt{x+2}\,dx + \frac{1}{2}\int \sqrt{x}\,dx \rightarrow \frac{1}{2}\cdot\frac{2}{3}x^{3/2} = \frac{1}{3}\sqrt{x^3}$$

Put $u^2 = x+2$,

$2u\,du = dx$, Then

$$\frac{1}{2}\int \sqrt{x+2}\,dx = \frac{1}{2}\int 2u^2\,du = \frac{1}{3}u^3, \text{ Then}$$

$$I = \boxed{\frac{1}{3}\left[(x+2)^{3/2} + x^{3/2}\right] + C}$$

Find:

(a) $I = \int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx$	(b) $J = \int \frac{x+7}{x^2-x-6}\,dx$	(c) $K = \int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}\,dx$
(d) $L = \int \frac{2x^5-5x}{(x^2+2)^2}\,dx$	(e) $M = \int \frac{x^2}{(x^2+1)^2}\,dx$	(f) $N = \int \sin^2 x \cos^2 x\,dx$

P.T.O

Ans:

(a) Using Partial fraction method; Then

$$I = \int \frac{5x^2+20x+6}{x(x^2+2x+1)}dx = \int \frac{5x^2+20x+6}{x(x+1)^2}dx = \int \left[\frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2} \right] dx$$

Then, $A(x+1)^2 + Bx(x+1) + Cx = 5x^2+20x+6$

Put: $x=-1 \Rightarrow -C = 5-20+6 \Rightarrow \boxed{C=9}$

$x = 0 \Rightarrow \boxed{A=6}$,

$x = 1 \Rightarrow 24 + 2B + 9 = 5+20+6 \Rightarrow \boxed{B=-1}$, Then

$$I = 6\ln|x| - \ln|x+1| + 9\frac{(x+1)^{-1}}{-1} + C$$

$$= \boxed{\ln\left|\frac{x^6}{x+1}\right| - \frac{9}{x+1} + C}$$

(b) $J = \int \frac{x+7}{x^2-x-6}dx$, using Partial fraction, Then

$$J = \int \frac{x+7}{(x+2)(x-3)}dx = \int \left[\frac{A}{x+2} + \frac{B}{x-3}\right] dx = A\ln|x+2| + B\ln|x-3| + C$$

$$= \boxed{\ln|(x+2)^A (x-3)^B| + C}$$

$A(x-3) + B(x+2) = x+7$

Put $x=3 \Rightarrow 5B = 10 \Rightarrow B = 2$,

$x = -2 \Rightarrow -5A = 5 \Rightarrow A = -1$;

Then $\boxed{J = \ln\left|\frac{(x-3)^2}{(x+2)}\right| + C}$

(c) $K = \int \frac{2x^2-4x-8}{x(x-1)(x^2+4)}dx$, using Partial fraction, Then

$$= \int \left[\frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+4}\right] dx$$

$$= \int \left[\frac{A}{x} + \frac{B}{x-1} + \frac{C}{2}\cdot\frac{2x}{x^2+4} + \frac{D}{x^2+4}\right] dx$$

$$= A\ln|x| + B\ln|x-1| + \frac{C}{2}\ln(x^2+4) + \frac{D}{2}\tan^{-1}\left(\frac{x}{2}\right) + C \rightarrow (1)$$

you should find the Constants A, B, C, D.

$$A(x-1)(x^2+4) + Bx(x^2+4) + (Cx+D)x(x-1) = 2x^3 - 4x - 8$$

Put $x = 0 \Rightarrow -4A = -8 \Rightarrow \boxed{A = 2}$

$x = 1 \Rightarrow 5B = -10 \Rightarrow \boxed{B = -2}$

$x = 2i \Rightarrow (2Ci + D)(-4 - 2i) = -16i - 8i - 8$

$$-8Ci - 4D + 4C - 2Di = -8 - 24i$$

$\therefore$ $4C - 4D = -8 \Rightarrow C - D = -2$

$-8C - 2D = -24 \Rightarrow 4C + D = 12$ } by adding

$5C = 10 \Rightarrow \boxed{C = 2}$, $\boxed{D = 4}$

$$\therefore \boxed{K = 2\tan^{-1}\left(\frac{x}{2}\right) + \ln\frac{x^2(x^2+4)}{(x-1)^2} + C} \ \#$$

(d) $L = \int \frac{2x^5 - 5x}{(x^2+2)^2}\,dx = \int \frac{2x^5 - 5x}{x^4 + 4x^2 + 4}\,dx$

لاحظ إن درجة البسط
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نقوم بقسمة البسط على المقام

$$\begin{array}{r|l} & 2x \\ \hline x^4 + 4x^2 + 4 & 2x^5 - 5x \\ & -(2x^5 + 8x^3 + 8x) \\ \hline & -8x^3 - 13x \end{array}$$

$$= \int 2x\,dx - \int \frac{x(8x^2+13)}{(x^2+2)^2}\,dx \to I$$

$$I = \int \left(\frac{Ax+B}{(x^2+2)} + \frac{Cx+D}{(x^2+2)^2}\right)dx$$

$$(Ax+B)(x^2+2) + (Cx+D) = x(8x^2+13)$$

Put $x = 0 \Rightarrow 2B + D = 0 \to ①$

$x = \sqrt{2}i \Rightarrow \sqrt{2}Ci + D = -3\sqrt{2}i \Rightarrow \boxed{D = 0}$, $\boxed{C = -3}$ sub. in ①, Then $\boxed{B = 0}$

$x = 1 \Rightarrow 3A - 3 = 21 \Rightarrow 3A = 24 \Rightarrow \boxed{A = 8}$

$$I = \int\left[\frac{8x}{x^2+2} - \frac{3x}{(x^2+2)^2}\right]dx = 4\int \frac{2x}{x^2+2}\,dx - \frac{3}{2}\int \frac{2x\,dx}{(x^2+2)^2}$$

Put $x^2 + 2 = u$, $2x\,dx = du$, Then

$$I = 4\ln(x^2+2) - \frac{3}{2}\int u^{-2}\,du$$

$$= 4\ln(x^2+2) + \frac{3}{2}u^{-1} + C \quad \text{sub. in } L\text{, Then}$$

$$L = \boxed{x^2 - 4\ln(x^2+2) - \frac{3}{2(x^2+2)} + C}$$

(e) $M = \int \frac{x^2+1-1}{(x^2+1)^2}\,dx = \int \frac{dx}{x^2+1} - \int \frac{dx}{(x^2+1)^2}$

$= \tan^{-1}x - \int \frac{dx}{(x^2+1)^2} \rightarrow$ Put $x = \tan\theta \Rightarrow dx = \sec^2\theta\,d\theta$

$= \int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta = \int \cos^2\theta\,d\theta = \frac{1}{2}\int(\cos 2\theta + 1)\,d\theta$

$= \frac{1}{4}\sin 2\theta + \theta/2 = \frac{1}{2}\sin\theta\cos\theta + \frac{\theta}{2} + C$

$= \frac{1}{2}\cdot\frac{1}{\sqrt{x^2+1}}\cdot\frac{x}{\sqrt{x^2+1}} + \frac{1}{2}\tan^{-1}x + C$

$\therefore \boxed{M = \frac{1}{2}\tan^{-1}x - \frac{x}{2(x^2+1)} + C}$ #

(f) $N = \int \sin^2x\,\cos^2x\,dx = \frac{1}{4}\int(1-\cos 2x)(1+\cos 2x)\,dx$

$= \frac{1}{4}\int(1-\cos^2 2x)\,dx = \frac{1}{4}\int \sin^2 2x\,dx = \frac{1}{8}\int(1-\cos 4x)\,dx$

$= \boxed{\frac{1}{8}x - \frac{1}{32}\sin 4x + C}$

Find: (a) $I = \int \sin^2x\,\cos^5x\,dx$. (d) $L = \int \sec^4 3x \cdot \tan^3 3x\,dx$.

(b) $J = \int \cos^4x\,dx$ (e) $M = \int \csc^4\frac{x}{2}\cdot\cot^4\frac{x}{2}\,dx$

(c) $K = \int \frac{\tan^3x}{\sqrt{\sec x}}\,dx$.

Ans:

(a) $I = \int \sin^2x\,\cos^4x\cdot\cos x\,dx = \int \sin^2x\,[1-\sin^2x]^2\cos x\,dx$

Put $u = \sin x \Rightarrow du = \cos x\,dx$, Then

$I = \int u^2(1-u^2)^2\,du = \int u^2(1-2u^2+u^4)\,du$

$= \int[u^2 - 2u^4 + u^6]\,du = \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$

$= \boxed{\frac{1}{3}\sin^3x - \frac{2}{5}\sin^5x + \frac{1}{7}\sin^7x + C}$ #

(b) $J = \int (\cos^2 x)^2 dx = \int \left[\frac{1}{2}(1+\cos 2x)\right]^2 dx$

$= \frac{1}{4}\int \left[1 + 2\cos 2x + \cos^2 2x\right] dx = \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}\int (1+\cos 4x)\,dx$

$= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\sin 4x + C$

$= \boxed{\frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C}$

(C) $\int \frac{\tan^3 x}{\sqrt{\sec x}} dx =$ $\Rightarrow$ Put $u = \sec x \Rightarrow du = \sec x \cdot \tan x\, dx$ then

$= \int \frac{\tan^2 x \cdot \tan x}{\sqrt{\sec x}} dx = \int \frac{(\sec^2 x - 1)\tan x\, dx}{\sqrt{\sec x}} = \int \frac{(u^2-1)}{\sqrt{u}} \cdot \frac{du}{u}$

$= \int (u^{\frac{1}{2}} - u^{-\frac{3}{2}})\,du = \frac{2}{3}u^{3/2} + 2u^{-\frac{1}{2}} + C = \boxed{\frac{2}{3}\sec^{3/2} x + \frac{2}{\sqrt{\sec x}} + C}$ #

(d) $= \int \sec^4 3x \cdot \tan^3 3x\, dx = \int \sec^2 3x\,(1+\tan^2 3x)\tan^3 3x\, dx$

$= \frac{3}{3}\int (\tan^3 3x \cdot \sec^2 3x + \tan^5 3x\, \sec^2 3x)\, dx$

Note: $\frac{d}{dx}(\tan ax) = a\sec^2 ax\, dx$

$= \boxed{\frac{1}{3}\left[\frac{1}{4}\tan^4 3x + \frac{1}{6}\tan^6 3x\right] + C}$

(e) $M = \int \csc^2\frac{x}{2}\,\cot^4\frac{x}{2}\,\csc^2\frac{x}{2}\cdot dx = \int (1+\cot^2\frac{x}{2})\cot^4\frac{x}{2}\,\csc^2\frac{x}{2}\,dx$

$= -2\int \frac{-1}{2}\cot^4\frac{x}{2}\,\csc^2\frac{x}{2}\,dx - 2\int \frac{-1}{2}\cot^6\frac{x}{2}\,\csc^2\frac{x}{2}\,dx$

Note: $\frac{d}{dx}(\cot ax) = -a\csc^2 ax$

$= \boxed{-2\left[\frac{1}{5}\cot^5\frac{x}{2} + \frac{1}{7}\cot^7\frac{x}{2}\right] + C}$